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ON THE PRECESSIONAL - SCREW MOTIONS OF A SOLID IMMERSED IN LIQUID*

V.N. RUBANOVSKII

The Kirchhoff-Klebsch problem on the inertial motion of a solid immersed in a liquid is considered. The precessional-screw motions of the solid consisting of two screw motions are investigated. The axis of one motion is fixed in space, and the other axis is fixed within the body. The necessary and sufficient kinematic conditions are given for the precessional-screw motions in differential and finite form. A method of finding such motions is given, their stability is studied and a geometrical interpretation of the body motions is presented.

1. Consider the problem of the inertial motion of a free solid bounded by a singly connected surface and containing multiconnected cavities completely filled with a perfect fluid in irrotational motion, in a perfect, homogeneous incompressible fluid unbounded in all directions. We will assume that the motion of the fluid outside the body caused by its motion is irrotational, and that the fluid is at rest at infinity.

The kinetic energy T of such a dynamic system can be written, apart from a constant determined by the periodic motion of the fluid within the body cavities, in the form /1/

$$T=\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(a_{ij}P_iP_j+b_{ij}R_iR_j+2c_{ij}P_iR_j),\quad a_{ij}=a_{ji},\ b_{ij}=b_{ji}$$

where a_{ij}, b_{ij}, c_{ij} are constants defined for the given system, while $R_1\ R_2, R_3$ and P_1, P_2, P_3 are projections of the impuslive force $\mathbf{R}$ and impulsive couple $\mathbf{P}$ of the system on the axes of a rectangular $Ox_1x_2x_3$ coordinate system rigidly bound to the body, neglecting the cyclic motion of the fluid within the body cavities.

Denoting by u_i and Ω_i the projections of the translational velocity $\mathbf{u}$ and instantaneous angular velocity $\mathbf{\Omega}$ of the body on the x_i axes, we obtain for them the following expressions:

$$u_1=\partial T/\partial R_1,\qquad \Omega_1=\partial T/\partial P_1\qquad (123) \tag{1.1}$$

The equations of motion of a body in a fluid have the form /1, 2/

$$d\mathbf{R}/dt+\mathbf{\Omega}\times\mathbf{R}=0,\quad d\mathbf{P}/dt+\mathbf{\Omega}\times(\mathbf{P}+\mathbf{k})+\mathbf{u}\times\mathbf{R}=0 \tag{1.2}$$

where $\mathbf{k}=(k_1, k_2, k_3)$ is the kinetic momentum vector of the cyclic motion of the fluid in the body cavities. Equations (1.2) admit of three first integrals

$$T=E=\text{const},\ \mathbf{R}^2=H^2=\text{const},\ (\mathbf{P}+\mathbf{k})\cdot\mathbf{R}=hH^2=\text{const} \tag{1.3}$$

Using the methods of screw calculus /3/ we introduce the impulsive $\mathbf{Q}=\mathbf{R}+\omega(\mathbf{P}+\mathbf{k})$ and kinematic $\mathbf{U}=\mathbf{\Omega}+\omega\mathbf{u}$ screw, where $\omega\ (\omega^2=0)$ is the Clifford number, and write (1.2) in the form of a single equation

$$d\mathbf{Q}/dt+\mathbf{U}\times\mathbf{Q}=0 \tag{1.4}$$

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From (1.4) it follows that the screw $\mathbf{Q}$ is fixed in space, and

$$\mathbf{Q}^2 = H^2 (1 + \omega h)^2 \tag{1.5}$$

Separating the principal part from the momentum part we obtain the last two integrals of (1.3).

2. We shall call the motion of a body in a fluid the precessional-screw (PS) motion if it consists of two screw motions the dual angle between whose axes remains constant and the axis of the first and second screw motion is fixed in space and the body respectively.

Below we shall only consider the case when the component screw motions are constant.

We shall show the necessary and sufficient kinematic conditions for the PS motion. Let Λ denote the unit screw of the axis fixed in space. We take x_3 as the axis fixed within the body, and denote its unit screw by $\mathbf{E}_3$. The dual angle between the axes of the screw Λ and $\mathbf{E}_3$ is denoted by $\Theta = \vartheta + \omega\vartheta^\circ$. The screws Λ and $\mathbf{E}_3$ satisfy the equations

$$d\Lambda/dt + \mathbf{U} \times \Lambda = 0, \quad \Lambda^2 = 1, \quad \Lambda \cdot \mathbf{E}_3 = \Lambda_3 = \cos\Theta \tag{2.1}$$

This yields the following equations in projections on the axes x_1, x_2, x_3:

$$\Lambda_1^{\cdot} = U_3\Lambda_2 - U_2\Lambda_3, \quad \Lambda_2^{\cdot} = U_1\Lambda_3 - U_3\Lambda_1 \tag{2.2}$$

$$U_1\Lambda_2 - U_2\Lambda_1 = 0 \tag{2.3}$$

Differentiating the integral relation (2.3) and taking (2.2) into account, we obtain an equation. Substituting into this equation the expressions

$$\Lambda_1 = \frac{U_1 \sin\Theta}{\sqrt{U_1^2 + U_2^2}}, \quad \Lambda_2 = \frac{U_2 \sin\Theta}{\sqrt{U_1^2 + U_2^2}}, \quad \Lambda_3 = \cos\Theta$$

obtained from (2.1) and (2.3), we arrive at the required condition

$$U_2 U_1^{\cdot} - U_1 U_2^{\cdot} - U_3 (U_1^2 + U_2^2) + (U_1^2 + U_2^2)^{3/2} \operatorname{ctg}\Theta = 0 \tag{2.4}$$

The relation (2.4) represents a screw analogue of the well-known kinematic Grioli condition of precessional motions of a solid with a fixed point /4/.

Let us introduce a rectangular coordinate system $O'\zeta_1\zeta_2\zeta_3$ whose ζ_3 axis coincides with the screw axis $\mathbf{Q} = H(1 + \omega h)\,\Gamma$ with unit screw Γ. The position of moving $Ox_1x_2x_3$ axes with respect to the fixed $O'\zeta_1\zeta_2\zeta_3$ axes is defined by the Euler angles Θ_*, Ψ_*, Φ_*. The projections U_i of the screw U on the x_i axes are connected with the dual Euler angles and their time derivatives by the relations /3/

$$\begin{aligned}
U_1 &= \Psi_*^{\cdot} \sin\Theta_* \sin\Phi_* + \Theta_*^{\cdot} \cos\Phi_* \\
U_2 &= \Psi_*^{\cdot} \sin\Theta_* \cos\Phi_* - \Theta_*^{\cdot} \sin\Phi_* \\
U_3 &= \Psi_*^{\cdot} \cos\Phi_* + \Phi_*^{\cdot}
\end{aligned}$$

Substituting these relations into (2.4), we reduce the condition of PS motions to the form

$$\begin{aligned}
&\Psi_*^{\cdot\cdot}\Theta_*^{\cdot}\sin\Theta_* - \Theta_*^{\cdot\cdot}\Psi_*^{\cdot}\sin\Theta_* + 2\Psi_*^{\cdot}\Theta_*^{\cdot 2}\cos\Theta_* + \\
&\Psi_*^{\cdot 3}\sin^2\Theta_*\cos\Theta_* = (\Theta_*^{\cdot 2} + \Psi_*^{\cdot 2}\sin^2\Theta_*)^{3/2}\operatorname{ctg}\Theta
\end{aligned} \tag{2.5}$$

The PS condition of motions can be written in a finite form in terms of the dual Euler angles. To do this, we shall consider the scalar product of two screw products

$$(\Lambda \times \Gamma)\cdot(\Gamma \times \mathbf{E}_3) = (\Lambda\cdot\Gamma)(\Gamma\cdot\mathbf{E}_3) - \Lambda\cdot\mathbf{E}_3$$

Since the scalar products of unit screws are equal to the cosines of the corresponding dual angles, the above expression yields $\sin K \sin\Theta_* \cos(\Psi_* - \Psi_{*0}) = \cos K \cos\Theta_* - \cos\Theta$, where $\Psi_* - \Psi_{*0}$ is the dual angle between the screws $(\Lambda \times \Gamma)\sin^{-1} K$ and $(\Gamma \times \mathbf{E}_3)\sin^{-1}\Theta_*$, Ψ_{*0} is the dual angle between the axis of the screw $(\Lambda \times \Gamma)\sin^{-1}K$ and the axis ζ_1, and K is the dual angle between the screws Λ and Γ ($K = \varkappa + \omega\varkappa^0$).

The last equation yields the required condition

$$\cos(\Psi_* - \Psi_{*0}) = (\cos K \cos\Theta_* - \cos\Theta)(\sin K \sin\Theta_*)^{-1} \tag{2.6}$$

which can be regarded as a general solution of the differential equation (2.5) with two arbitrary dual constants Ψ_{*0} and K. The relation is an analogue of the condition of precessional motion of a body with a fixed point* (*Gorr G.V. Methods of constructing complete solutions of rigid body dynamics. Doctorate Dissertation, Moscow, MGU, 1982.)

3. We shall now describe the method of finding the PS motions. Multiplying both sides of the (2.1) scalarly by the screw $\mathbf{E}_3$, we obtain the relation $\mathbf{E}_3\cdot(\mathbf{U} \times \Lambda) = 0$ from which it follows that the screw $\mathbf{U}$ can be written in the form

$$\mathbf{U} = \Psi^{\cdot}\Lambda + \Phi^{\cdot}\mathbf{E}_3 \tag{3.1}$$

where Ψ is the dual angle between the axes of the screws $\Lambda \times (\Gamma \times \Lambda)$ and $\mathbf{E}_3 \times \Lambda$, and

Φ is the dual angle between the axis of the screw $\mathbf{E}_3 \times \mathbf{\Lambda}$ and the x_1 axis.

Subsituting (3.1) into (2.1), we obtain the following equation for determining Λ:

$$\mathbf{\Lambda}^{\cdot} + \Phi^{\cdot} (\mathbf{E}_3 \times \mathbf{\Lambda}) = 0, \quad \mathbf{\Lambda}^2 = 1 \tag{3.2}$$

Integrating (3.2) we find the following expression for the rectangular dual coordinate $\Lambda_i = \lambda_i + \omega \lambda_i^\circ$ $(i = 1, 2, 3)$ of the screw $\mathbf{\Lambda}$:

$$\Lambda_1 = \sin \Theta \sin \Phi, \ \Lambda_2 = \sin \Theta \cos \Phi, \ \Lambda_3 = \cos \Theta, \ \Phi = \Phi^{\cdot} t + \mathrm{A} \tag{3.3}$$

where $\mathrm{A} = \alpha + \omega \alpha^\circ$ is an arbitrary dual constant.

Expressions (3.1) and (3.3) yield the following expressions for the coordinates U_i of the screw $\mathbf{U}$:

$$\begin{aligned} U_1 &= \Omega_1 + \omega u_1 = \Psi^{\cdot} \sin \Theta \sin \Phi \\ U_2 &= \Omega_2 + \omega u_2 = \Psi^{\cdot} \sin \Theta \cos \Phi \\ U_3 &= \Omega_3 + \omega u_3 = \Psi^{\cdot} \cos \Theta + \Phi^{\cdot} \end{aligned} \tag{3.4}$$

Separating the principal and momentum parts, we obtain

$$\Omega_1 = \psi^{\cdot} \sin \vartheta \sin \varphi, \quad \Omega_2 = \psi^{\cdot} \sin \vartheta \cos \varphi, \ \Omega_3 = \psi^{\cdot} \cos \vartheta + \varphi^{\cdot} \tag{3.5}$$

$$\varphi = \varphi^{\cdot} t + \alpha$$

$$\begin{aligned} u_1 &= (\psi^{\cdot\circ} \sin \vartheta + \psi^{\cdot} \vartheta^\circ \cos \vartheta) \sin \varphi + \psi^{\cdot} \ (\varphi^{\cdot\circ} t + \alpha^\circ) \sin \vartheta \cos \varphi \\ u_2 &= (\psi^{\cdot\circ} \sin \vartheta + \psi^{\cdot} \vartheta^\circ \cos \vartheta) \cos \varphi - \psi^{\cdot} \ (\varphi^{\cdot\circ} t + \alpha^\circ) \sin \vartheta \sin \varphi \\ u_3 &= \psi^{\cdot\circ} \cos \vartheta - \psi^{\cdot} \vartheta^\circ \sin \vartheta + \varphi^{\cdot\circ} \end{aligned} \tag{3.6}$$

To find the screw $\mathbf{\Gamma}$ we use the easily confirmed expansion

$$\mathbf{\Gamma} = \left(\cos \mathrm{K} + \frac{\cos \Theta \sin \mathrm{K} \sin \Psi}{\sin \Theta}\right) \mathbf{\Lambda} - \frac{\sin \mathrm{K} \sin \Psi}{\sin \Theta} \mathbf{E}_3 - \frac{\sin \mathrm{K} \cos \Psi}{\sin \Theta} (\mathbf{\Lambda} \times \mathbf{E}_3) \tag{3.7}$$

from which we obtain

$$\begin{aligned} \Gamma_1 &= (\sin \Theta \cos \mathrm{K} + \cos \Theta \sin \mathrm{K} \sin \Psi) \sin \Phi - \sin \mathrm{K} \cos \Psi \cos \Phi \\ \Gamma_2 &= (\sin \Theta \cos \mathrm{K} + \cos \Theta \sin \mathrm{K} \sin \Psi) \cos \Phi + \sin \mathrm{K} \cos \Psi \sin \Phi \\ \Gamma_3 &= \cos \Theta \cos \mathrm{K} - \sin \Theta \sin \mathrm{K} \sin \Psi, \quad \Psi = \Psi^{\cdot} t + \mathrm{B} \\ \mathrm{B} &= \beta + \omega \beta^\circ = \mathrm{const} \end{aligned}$$

for the coordinates $\Gamma_i = \gamma_i + \omega \gamma_i^\circ$ $(i = 1, 2, 3)$ of the screw $\mathbf{\Gamma}$.

Separating again the principal and momentum parts we obtain

$$\begin{aligned} \gamma_1 &= (\sin \vartheta \cos \varkappa + \cos \vartheta \sin \varkappa \sin \psi) \sin \varphi - \\ &\quad \sin \varkappa \cos \psi \cos \varphi \\ \gamma_2 &= (\sin \vartheta \cos \varkappa + \cos \vartheta \sin \varkappa \sin \psi) \cos \varphi + \\ &\quad \sin \varkappa \cos \psi \sin \varphi \\ \gamma_3 &= \cos \vartheta \cos \varkappa - \sin \vartheta \sin \varkappa \sin \psi \end{aligned} \tag{3.8}$$

$$\begin{aligned} \gamma_1^\circ &= (\vartheta^\circ \cos \vartheta \cos \varkappa - \varkappa^\circ \sin \vartheta \sin \varkappa) \sin \varphi - (\vartheta^\circ \sin \vartheta \sin \varkappa - \\ &\quad \varkappa^\circ \cos \vartheta \cos \varkappa) \sin \varphi \sin \psi - \varkappa^\circ \cos \varkappa \cos \varphi \cos \psi + \\ &\quad (\varphi^\circ + \psi^\circ \cos \vartheta) \sin \varkappa \sin \varphi \cos \psi + \varphi^\circ \sin \vartheta \cos \varkappa \cos \varphi + \\ &\quad (\psi^\circ + \varphi^\circ \cos \vartheta) \sin \varkappa \cos \varphi \sin \psi \\ \gamma_2^\circ &= (\vartheta^\circ \cos \vartheta \cos \varkappa - \varkappa^\circ \sin \vartheta \sin \varkappa) \cos \varphi - (\vartheta^\circ \sin \vartheta \sin \varkappa - \\ &\quad \varkappa^\circ \cos \vartheta \cos \varkappa) \cos \varphi \sin \psi + \varkappa^\circ \cos \varkappa \sin \varphi \cos \psi + \\ &\quad (\varphi^\circ + \psi^\circ \cos \vartheta) \sin \varkappa \cos \varphi \cos \psi - \varphi^\circ \sin \vartheta \cos \varkappa \sin \varphi - \\ &\quad (\psi^\circ + \varphi^\circ \cos \vartheta) \sin \varkappa \sin \varphi \sin \psi \\ \gamma_3^\circ &= -(\vartheta^\circ \sin \vartheta \cos \varkappa + \varkappa^\circ \cos \vartheta \sin \varkappa) - (\vartheta^\circ \cos \vartheta \sin \varkappa + \\ &\quad \varkappa^\circ \sin \vartheta \cos \varkappa) \sin \psi - \psi^\circ \sin \vartheta \sin \varkappa \cos \psi \end{aligned} \tag{3.9}$$

Expressions for R_i, P_i $(i = 1, 2, 3)$ are obtained from the relation

$$\mathbf{R} + \omega (\mathbf{P} + \mathbf{k}) = H (1 + \omega h)(\gamma + \omega \gamma^\circ)$$

by writing it in terms of the coordinates and separating the principal and momentum parts

$$R_1 = H \gamma_1, \qquad P_1 = h R_1 + H \gamma_1^\circ - k_1 \qquad (123) \tag{3.10}$$

On the other hand, we have the relations (1.1) for Ω_i, u_i $(i = 1, 2, 3)$. We write these relations in the form (the superscript T denotes transposition)

$$\mathbf{\Omega} = \mathbf{A} \cdot P + C \cdot \mathbf{R}, \quad \mathbf{u} = C^T \cdot P + \mathbf{B} \cdot \mathbf{R}, \quad \mathbf{A} = \| a_{ij} \|_1^3,$$
$$\mathbf{B} = \| b_{ij} \|_1^3, \quad \mathbf{C} = \| c_{ij} \|_1^3$$

Substituting here the expressions for $\mathbf{R}$ and $\mathbf{P}$ from (3.10), we obtain

$$\Omega = H\mathbf{A}\cdot\gamma^\circ + H(\mathbf{C} + h\mathbf{A})\cdot\gamma - \mathbf{A}\cdot\mathbf{k} \quad (3.11)$$

$$\mathbf{u} = H\mathbf{C}^T\cdot\gamma^\circ + H(\mathbf{B} + h\mathbf{C}^T)\cdot\gamma - \mathbf{C}^T\cdot\mathbf{k} \quad (3.12)$$

Identifying now the expressions for Ω and $\mathbf{u}$ given by (3.5), (3.11) and (3.6), (3.12), we obtain the condtions for the PS motions of the body to exist.

Let us consider relation (3.11).Its left-hand side contains no secular terms. The necessary condition for its right-hand side also to contain no secular terms is, that Eqs.(3.9) for $\gamma_1^\circ, \gamma_2^\circ, \gamma_3^\circ$ should contain no secular terms. This leads to the conditions

$$\psi^{\cdot\circ}\sin\vartheta\sin\varkappa = 0, \qquad \varphi^{\cdot\circ}\sin\vartheta\cos\varkappa = 0$$
$$(\varphi^{\cdot\circ} + \psi^{\cdot\circ}\cos\vartheta)\sin\varkappa = 0, \qquad (\psi^{\cdot\circ} + \varphi^{\cdot\circ}\cos\vartheta)\sin\varkappa = 0$$

which hold for the following cases:

$$\text{a) } \varkappa = \varphi^{\cdot\circ} = 0;\ \text{ b) } \vartheta = \varphi^{\cdot\circ} + \psi^{\cdot\circ} = 0;\ \text{ c) } \vartheta = \pi,\ \ \varphi^{\cdot\circ} = \psi^{\cdot\circ};\ \text{ d) } \psi^{\cdot\circ} = \varphi^{\cdot\circ} = 0 \quad (3.13)$$

Let us investigate the PS motions in case a) of (3.13). We shall assume for simplicity that $\varkappa^\circ = 0$, in which case the axes of the screws Λ and Γ will coincide.

Identifying the expressions (3.5), (3.11) for Ω_i and (3.6), (3.12) for u_i $(i = 1, 2, 3)$, we obtain the following nine groups of conditions for the PS motions to exist:

$$[a_{11}(h\sin\vartheta + \vartheta^\circ\cos\vartheta) - a_{12}\alpha^\circ\sin\vartheta + c_{11}\sin\vartheta]H = \delta_1^1\psi^\cdot\sin\vartheta \quad (123)$$

$$[a_{11}\alpha^\circ\sin\vartheta + a_{12}(h\sin\vartheta + \vartheta^\circ\cos\vartheta) + c_{12}\sin\vartheta]H = \delta_1^2\psi^\cdot\sin\vartheta \quad (123)$$

$$[a_{13}(h\cos\vartheta - \vartheta^\circ\sin\vartheta) + c_{13}\cos\vartheta]H - a_{11}k_1 - a_{12}k_2 - a_{13}k_3 = \delta_1^3(\varphi^\cdot + \psi^\cdot\cos\vartheta) \quad (123)$$

$$a_{12}\varphi^{\cdot\circ}H\sin\vartheta = 0\ (123),\quad a_{11}\varphi^\circ H\sin\vartheta = 0 \quad (123)$$

$$[c_{11}(h\sin\vartheta + \vartheta^\circ\cos\vartheta) - c_{21}\alpha^\circ\sin\vartheta + b_{11}\sin\vartheta]H = g_1 \quad (123)'$$

$$[c_{11}\alpha^\circ\sin\vartheta + c_{21}(h\sin\vartheta + \vartheta^\circ\cos\vartheta) + b_{12}\sin\vartheta]H = g_1' \quad (123)'$$

$$[c_{31}(h\cos\vartheta - \vartheta^\circ\sin\vartheta) + b_{13}\cos\vartheta]H - c_{11}k_1 - c_{21}k_2 - c_{31}k_3 = 0 \quad (123)'$$

$$c_{21}\varphi^{\cdot\circ}H\sin\vartheta = \delta_1^2\psi^\cdot\varphi^{\cdot\circ}\sin\vartheta\ \ (123)',\quad c_{11}\varphi^{\cdot\circ}H\sin\vartheta = \delta_1^1\psi^\cdot\varphi^{\cdot\circ}\sin\vartheta \quad (123)'$$

$$g_1 = g_2' = \psi^{\cdot\circ}\sin\vartheta + \psi^\cdot\vartheta^\circ\cos\vartheta,\ \ g_2 = -g_1' = -\psi^\cdot\alpha^\circ\sin\vartheta,\ \ g_3 = g_3' = 0$$

Here δ_j^i is the Kronecker delta, the symbol (123) means that another two analogous relations are obtained from the relation shown by cyclic permutation of the indices 1, 2, 3 accompanying g_i, g_i', of the first index of a_{ij}, b_{ij}, c_{ij} and of the lower index of δ_j^i; and the prime accompanying this symbol means that the second index of c_{ij} should be interchanged in the manner shown.

In the course of analysing the above conditions we can assume that $a_{12} = 0$, $c_{12} = c_{21}$, otherwise it can be achieved by a oorresponding choice of the $Ox_1x_2x_3$ coordinate system. Analysis of these conditions leads to the following relations:

$$k_1 = k_2 = 0 \quad (3.14)$$

$$a_{12} = c_{12} = c_{21} = b_{12} = 0,\ c_{31} = a_{13}\nu,\ c_{32} = a_{23}\nu,\ c_{13} = a_{13}\mu \quad (3.15)$$
$$c_{23} = a_{23}\mu,\quad c_{ii} = c + a_{ii}\nu,\ b_{ii} = b + a_{ii}\nu^2\ \ (i = 1,\ 2)$$
$$c_{33} = c + a_{33}\mu + c',\ b_{13} = a_{13}\nu\mu,\ b_{23} = a_{23}\nu\mu$$

$$\alpha^\circ = \beta^\circ = \varphi^{\cdot\circ} = 0,\quad \varphi^\cdot = c'H\cos\vartheta, \psi^\cdot = cH, \psi^{\cdot\circ} = (b + ch)H \quad (3.16)$$

$$\vartheta^\circ = {}^1/_2[\mu - \nu + c^{-1}(b - b_{33} + a_{33}\mu^2 + c'\mu)]\sin 2\vartheta \quad (3.17)$$

$$H = ck_3(b_{33} - b - a_{33}\mu^2 - c'\mu)^{-1}\cos^{-1}\vartheta \quad (3.18)$$
$$h = -\nu + [\nu - \mu + c^{-1}(b_{33} - b - a_{33}\mu^2 - c'\mu)]\cos^2\vartheta$$
$$2E = [(b - 2c\nu)\sin^2\vartheta + (b_{33} - a_{33}\mu^2 - 2c\mu - 2c'\mu)\times\cos^2\vartheta)]H^2$$

where $a_{11}, a_{22}, a_{33}, a_{13}, a_{23}, b_{33}, b, c, c', \nu, \mu, k_3, \vartheta$ $(b > 2c\nu,\ b_{33} > a_{33}\mu^2 + 2(c + c')\mu)$ are independent real parameters and $b_{33} = b + a_{33}\mu^2 + c'\mu$ if $k_3 = 0$.

When the relations (3.14)–(3.18) all hold, Eqs.(1.2) admit of the following solutions:

$$R_1 = H\sin\vartheta\sin(\varphi^\cdot t + \alpha),\ \ P_1 = H(h\sin\vartheta + \vartheta^\circ\cos\vartheta)\sin(\varphi^\cdot t + \alpha) \quad (3.19)$$
$$R_2 = H\sin\vartheta\cos(\varphi^\cdot t + \alpha),\ \ P_2 = H(h\sin\vartheta + \vartheta^\circ\cos\vartheta)\cos(\varphi^\cdot t + \alpha)$$
$$R_3 = H\cos\vartheta,\ \ P_3 = H(h\cos\vartheta - \vartheta^\circ\sin\vartheta) - k_3$$

for which Ω_i, u_i have the following expressions:

$$\Omega_1 = \psi^{\cdot} \sin\vartheta \sin(\varphi^{\cdot} t + \alpha), \quad u_1 = (\psi^{\cdot\circ} \sin\vartheta + \psi^{\cdot}\vartheta^{\circ} \cos\vartheta) \times \sin(\varphi^{\cdot} t + \alpha) \tag{3.20}$$

$$\Omega_2 = \psi^{\cdot} \sin\vartheta \cos(\varphi^{\cdot} t + \alpha), \quad u_2 = (\psi^{\cdot\circ} \sin\vartheta + \psi^{\cdot}\vartheta^{\circ} \cos\vartheta) \times \cos(\varphi^{\cdot} t + \alpha)$$

$$\Omega_3 = \psi^{\cdot} \cos\vartheta + \varphi^{\cdot}, \quad u_3 = \psi^{\cdot\circ} \cos\vartheta - \psi^{\cdot}\vartheta^{\circ} \cos\vartheta$$

Equations (3.14) show that in the PS motion the hydrostatic moment vector of the internal cyclic motions is parallel to the x_3 axis of the inherent screw motion of the body. Equations (3.15) impose the corresponding constraints on the form of the outer surface of the body and the distribution of its mass. Equations (3.16) determine the values of the kinematic characteristics of the PS motion. Equation (3.17) shows that when the angle ϑ is varied and the remaining parameters are kept fixed, the x_3 axis describes a cylindroid. Expressions (3.18) give the values of the constants of the first three integrals of (1.3) for the PS motions of the body.

4. Next we shall give a geometrical interpretation of the motion of the body described by (3.20). To do this we write (3.1), taking (3.16) into account, in the form

$$\Omega e^{\omega P}\mathbf{E} = \varphi^{\cdot}\mathbf{E}_3 + \psi^{\cdot} e^{\omega p}\mathbf{\Gamma}, \quad \mathbf{U} = \Omega e^{\omega P}\mathbf{E}, \quad p = \psi^{\cdot\circ}/\psi^{\cdot} \tag{4.1}$$

where Ω, P, $\mathbf{E}$ is the modulus of the vector, a parameter and the unit vector of the screw $\mathbf{U}$, and p is a parameter of the screw $\Psi\mathbf{\Gamma}$.

Squaring both sides of (4.1) and separating the principal and momentum parts, we find

$$\Omega^2 = \varphi^{\cdot 2} + \psi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\cos\vartheta, \quad P = \frac{(\psi^{\cdot} + \varphi^{\cdot}\cos\vartheta)\psi^{\cdot\circ} - \varphi^{\cdot}\psi^{\cdot}\vartheta^{\circ}\sin\vartheta}{\varphi^{\cdot 2} + \psi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\cos\vartheta} \tag{4.2}$$

Now, scalar mutliplying (4.1) by the screws $\mathbf{E}$ and $\mathbf{\Gamma}$, and denoting by $D = \delta + \omega\delta^{\circ}$ and $\Theta - D = \vartheta - \delta + \omega(\vartheta^{\circ} - \delta^{\circ})$ the dual angles formed by the screw $\mathbf{E}$ with the screws $\mathbf{E}_3$ and $\mathbf{\Gamma}$, we obtain

$$\Omega e^{\omega P}\cos D = \varphi^{\cdot} + \psi^{\cdot} e^{\omega p}\cos\Theta, \quad \Omega e^{\omega P}\cos(\Theta - D) = \varphi^{\cdot}\cos\Theta + \psi^{\cdot} e^{\omega p}$$

Separating the principal and momentum parts, we obtain

$$\cos\delta = (\varphi^{\cdot} + \psi^{\cdot}\cos\vartheta)\Omega^{-1}, \quad \delta^{\circ} = [\varphi^{\cdot}\psi^{\cdot\circ}\sin\vartheta + \psi^{\cdot}(\psi^{\cdot} + \varphi^{\cdot}\cos\vartheta)\vartheta^{\circ}]\Omega^{-2} \tag{4.3}$$

$$\cos(\vartheta - \delta) = (\psi^{\cdot} + \varphi^{\cdot}\cos\vartheta)\Omega^{-1}, \quad \vartheta^{\circ} - \delta^{\circ} = [-\varphi^{\cdot}\psi^{\cdot\circ}\sin\vartheta + \varphi^{\cdot}(\varphi^{\cdot} + \psi^{\cdot}\cos\vartheta)\vartheta^{\circ}]\Omega^{-2}$$

The above formulas show that the dual angles formed by the unit vector $\mathbf{E}$ of the screw $\mathbf{U}$ with the screws $\mathbf{E}_3$ and $\mathbf{\Gamma}$, remain constant during the motion of the body. Therefore the axis of the screw $\mathbf{U}$ will describe, during the motion, single sheet hyperboloids within the body and in the static space

$$x_1^2 + x_2^2 - x_3^2 \operatorname{tg}^2\delta = \delta^{\circ 2}, \quad \xi_1^2 + \xi_2^2 - \xi_3^2 \operatorname{tg}^2(\vartheta - \delta) = (\vartheta^{\circ} - \delta^{\circ})^2 \tag{4.4}$$

Here the fixed coordinate system $O''\xi_1\xi_2\xi_3$ is chosen so that the ξ_3 axis is directed along the axis of the impulsive screw $\mathbf{Q}$, and the points of intersection of the axis of dual angle Θ with the axes of the screws $\mathbf{Q}$ and $\mathbf{E}_3$ are taken as the origin of coordinates O, O'' of the moving and fixed coordinate axes.

The motion of the body in a fluid described by the solution (3.20) can be regarded as the result of rolling a moving hyperboloid over the fixed hyperboloid about the common generatrix with angular velocity Ω, and its slippage along this generatrix with velocity $P\Omega$. When $c_{ij} \to 0, b_{ij} \to 0$, the problem of the motion of the body in a fluid becomes a problem of the motion of a gyrostat in a state of equilibrium, and the formulas (3.14)–(3.17) lead to the well-known results

$$a_{11} = a_{22}, \; a_{13} = a_{23} = 0, \; k_1 = k_2 = 0$$

$$(C - A)\psi^{\cdot}\cos\vartheta + C\varphi^{\cdot} + k_3 = 0, \; \psi^{\cdot\circ} = \vartheta^{\circ} = 0, \; A = a_{11}^{-1}$$

$$C = a_{33}^{-1}$$

while (4.2) and (4.3) together yield

$$\delta^{\circ} = \vartheta^{\circ} = 0, \quad \operatorname{tg}\delta = \frac{\psi^{\cdot}\sin\vartheta}{\varphi^{\cdot} + \psi^{\cdot}\cos\vartheta}, \quad \operatorname{tg}(\vartheta - \delta) = \frac{\varphi^{\cdot}\sin\vartheta}{\psi^{\cdot} + \varphi^{\cdot}\cos\vartheta}$$

$$x_1^2 + x_2^2 - x_3^2 \operatorname{tg}^2\delta = 0, \quad \xi_1^2 + \xi_2^2 - \xi_3^2 \operatorname{tg}^2(\vartheta - \delta) = 0$$

In this case the body moves in such a manner that the moving asymptotic cone rolls without slippage along the fixed asymptotic cone with angular velocity Ω, and the tops of the cones overlap.

5. We shall study the stability of the motion (3.19) in the case when

$$a_{11} = a_{22}, \quad a_{13} = a_{23} = 0 \tag{5.1}$$

When conditions (5.1) hold, the doubled kinetic energy of the system is given, taking

(3.15) taken into account, by the expression

$$2T = a_{11}(P_1^2 + P_2^2) + a_{33}P_3^2 + b_{11}(R_1^2 + R_2^2) + b_{33}R_3^2 + 2c_{11}(P_1R_1 + P_2R_2) + 2c_{33}P_3R_3 \tag{5.2}$$

$$c_{11} = c + a_{11}\nu, \quad c_{33} = c + a_{33}\mu + c', \quad b_{11} = b + a_{11}\nu^2$$

This is the form of kinetic energy of a body whose form and distribution of mass remain unchanged when the body rotates about the x_3 axis by an angle $\pi/2$. A four-bladed ship propeller is an example of such a body.

When conditions (5.1) hold, Eqs.(1.2) admit of the first integral $P_3 = \text{const}$. Let us write expression (5.2) taking (3.15) and integrals (1.3) into account, in the form

$$2T = a_{11}[(P_1 + \nu R_1)^2 + (P_2 + \nu R_2)^2] + a_{33}P_3^2 + 2(c' + a_{33}\mu)P_3R_3 + (b_{33} - b)R_3^2 - 2ck_3R_3 + (b + 2ch)H^2 = \text{const} \tag{5.3}$$

Let us find the values of the variables $L_i = P_i + \nu R_i$ $(i = 1, 2)$ for which T is constant under the condition that P_3 be regarded as a parameter. From (5.3) we obtain

$$\frac{\partial T}{\partial L_i} = a_{ii}L_i = 0 \ (i = 1, 2), \quad \frac{\partial T}{\partial R_3} = (c' + a_{33}\mu)P_3 + (b_{33} - b)R_3 - ck_3 = 0 \tag{5.4}$$

We have for the solution (3.19)

$$L_1 = P_1 + \nu R_1 = 0, \quad L_2 = P_2 + \nu R_2 = 0, \quad P_3 + \mu R_3 = 0$$

Substituting into (5.4) $P_3 = -\mu R_3$ we obtain from these equations

$$L_1 = P_1 + \nu R_1 = 0, \ L_2 = P_2 + \nu R_2 = 0, \ R_3 = ck_3(b_{33} - b - a_{33}\mu^2 - c'\mu)^{-1} \tag{5.5}$$

The above expressions for L_1, L_2, R_3 agree with the corresponding expressions obtained from (3.19). This implies that the integral $T = \text{const}$ has a stationary value for the solution (3.19), provided that it is regarded as a function of the variables L_1, L_2, R_3, with P_3 regarded as a parameter.

The demand that the second variation be positive definite

$$\delta^2 T = a_{11}[(\delta L_1)^2 + (\delta L_2)^2] + (b_{33} - b)(\delta R_3)^2$$

leads to the inequality $b_{33} > b$ which, by virtue of the Routh theorem, represents the sufficient condition of stability of the PS motions (3.19) relative to the quantities L_1, L_2, P_3, R_3, for which relations (5.1) hold.

We note that in the case of a balanced gyrostat $c_{ij} = b_{ij} = 0 \ (i, j = 1, 2, 3)$, therefore, taking into account (3.15) we have $b = -a_{11}\nu^2 < 0$ and conditions $b_{33} > b$ holds.

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